Geometry of the sacrifice in the śulbasūtra[1] of Vedic India

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Summary

- Geometrisation of the sacrifice : units (p.2), instruments (p.4), constructions (p.4), extensions (p.9).
- "Fundamentals" of the Vedic geometry (p.12) : orientation (p.12), construction of a square (p.13), theorem of the diagonal of the rectangle (p.15), adding and subtracting two squares (p.20), transforming a rectangle into a square with the same area and vice versa (p.20), transforming a square into a circle with the same area and vice-versa (p.24).
- Analogies with some problems of Euclid's *Elements* (p.25).
- Conclusion : an embryonic corpus of geometry (p.32).
- References (p.35).

Geometrisation of the sacrifice

In the Vedic literature[2], the energy which has to be kept and freed by the sacrifice, is expressed in expansion, an omnipresent leitmotiv: " This bull here, *by swelling*

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1 Extract from *La figure et le monde. Une archéologie de la géométrie. Peuples paysans sans écriture et premières civilisations*. Olivier Keller. Paris, Vuibert, 2006. Pages 139-168. Translated by Gilles Schaufelberger and Helen Goethals. © Vuibert.

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2 We will use the following abreviations: *M. Śulb.*, *B. Śulb.*, *Ā. Śulb.* and *K. Śulb.* respectively for Mānava, Baudhāyana, Āpastamba et Katyāyana Śulbasūtra. Sen and Bag, 1983.

itself, since it is Indra, may give us wealth and wisdom "[3], "Prajāpati, in the embryo, is very busy; unbegotten, he is born and *spreads* in many ways "[4], " This god, by whom the robust sky and the earth were set, by whom the sun was propped up, propped up the celestial vault, *this god who in the median space measures the expanse* "[5], " Vaiśvānara, *who measured out the realms of air*, Sage very wise who made the lucid spheres of heaven, The Undeceivable *who spread out all the worlds*, keeper is he and guard of immortality"[6], "Varuṇa, imperial Ruler, Who hath struck out, like one who slays the victim, earth *as a skin to spread* in front of Sūrya. … I will declare this mighty deed of magic, of glorious Varuṇa the Lord Immortal,
Who standing in the firmament hath meted the earth out with the Sun as with a measure"[7]. Prajāpati encourages himself with magnificent words:

> "Now this Person Prajjāpati desired, *'May I be more (than one), may I be reproduced!'* He toiled, he practised austerity. Being worn out with toil and austerity, he created first of all the Brahman, the triple science … 'This (earth) has indeed become a foundation!' (he thought): hence it became the earth. *He spread it out*, and it became the broad one (or earth). And she (the earth), thinking herself quite perfect, sang; and inasmuch as she sang, therefore she is Gâyatrî. But they also say, 'It was Agni, indeed, on her (the earth's) back, who thinking himself quite perfect, sang; and inasmuch as he sang, therefore Agni is Gâyatra.' And hence whosoever thinks himself quite perfect, either sings or delights in song."[8].

The resulting geometry is particularly promising, because, as a translation of the primordial sacrifice, everything must come from the sacrificer-demiurge. From him, the measure and the form must occur, *he is in charge of the units and the building*.

Units

The fundamental length unit, the *puruṣa*, a term meaning "man", is the height of the sacrificer with raised arms. So reads the theory, but in practice, there is a trend towards standardisation. Mānava gives objective units, starting from mustard seeds, in case the sacrificer would be "diseased or short from birth".

3 *Atharva Veda,* 9, 4, From the French transl. by J. Varenne, 1984.

4 *Taittiriya Aranyaka* 10.1. Same source.

5 -id-

6 *Ṛg Veda* 6-7 Transl. Ralph Griffith 1896. http://www.sacred-texts.com

7 *Ṛg Veda* 5-85 Transl. Ralph Griffith 1896. http://www.sacred-texts.com

8 *Śatapatha Brāhmaṇa* 10-1-1. Translation Julius Eggeling, 1894. http://www.sacred-texts.com

Most common units according to Baudhāyana
1 *puruṣa* = height of the sacrificer with raised arms = 120 *aṅgulas*
1 *pradeśa* = 12 *aṅgulas*
1 *aratni* = 2 *pradeśas* = 24 *aṅgulas*
1 *prakrama* = 30 *aṅgulas*

Normally, " 120 *aṅgulas* of a man always equal his 5 *aratnis* or 10 *padas*; depending on the man's stature, the measure can be smaller or larger[9] ". Baudhāyana gives first objective measures, on the basis of seeds, but he points out further that the bamboo pierced with three holes has the same height as "a man with uplifted arms[10] ", i.e. a *puruṣa*. Āpastamba gives only a human definition, excluding all others, without taking into account any possible deformity of the sacrificer: " According to the tradition, to be measured with a *puruṣa* means to be measured with a bamboo's rod. Two holes are made at the ends of a bamboo rod at a distance equal to the height of the sacrificer with uplifted hands, and a third hole is made at the middle[11] ". He gives then the construction of the 1 *puruṣa* sided square, using this tool.

Area units have no specific names; a *puruṣa*, for example, could mean a square *puruṣa*, according to the context, with the underlying idea that a square is produced by its side. Āpastamba says for example:

> " Twice the measure produces four; thrice the measure, nine ... a cord 1 1/2 *puruṣa* long makes 2 1/4, a cord of 2 1/2 *puruṣas*, 6 1/4 ... With half the side of a square, a square one-fourth in area is produced, because four such squares to complete the area are produced with twice the half side ".

Āpastamba's sūtra 3-9 shows how the calculation of an area leads to represent what we would write down today as $(a + b)^2 = a^2 + 2ab = b^2$: " Now the method when the side is increased. With the side and the length by which the side is increased is drawn which is placed on either side. A square is formed with the length by which the side is increased and placed in the corner ". The reader understands easily that if *a* " produces " the initial square, and if *b* is the increase of the side, two rectangles of

9 M. Śulb., 11-7.

10 B. Śulb. 8-9.

11 Ā. Śulb. 8-7 and 9-1.

an area *ab* are drawn, and a final square with a side equal to *b* is put in the corner to complete the final square (fig. 1)

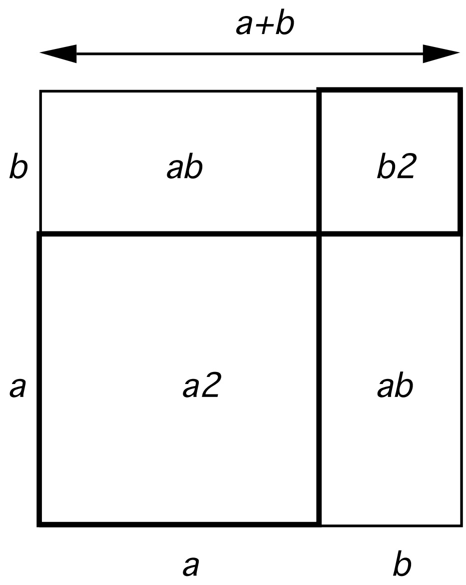

Figure 1. Figuration of the calculation of the square produced by a side *a* when it is increased by *b*

The volumes are mentioned only by Mānava (10-9), with no further development: " Multiply the length by the breadth separately and that again by the height: this always gives the result in cubic measures ".

Instruments

The author of the *Ṛg Veda* remained perplexed by the question of the creation from nothingness. Here, the *Śulbasūtras* try to move closer to it, if we can say so, by reducing the construction instruments to a minimum. The authorized instruments are a cord, a bamboo pierced with two holes at its ends and one in the middle, and stakes. The cord is the fundamental instrument; it must be deprived of knots. In fact, the work is done with several cords, essentially to build right-angled triangles. Cords can be associated and marked in certain places, according to the needs; but in no way can we speak of graduated cords.

Constructions

The vital energy the sacrifice must recreate and release consists mainly of an area equal to 7.5 squares of 1 *puruṣa* side, that we will note down as 7.5 p^2. We have already emphasised the importance of the number seven and given its origin (see footnote 23): it is associated here with Prajāpati as " seven-figure ", who the *Śatapatha Brāhmaṇa* introduces thus: in the beginning, there was the non-being, or prophets, or vital breaths[12], and they created seven figures. To be able to procreate, these seven figures had to merge into only one, Prajāpati:

> "That same Person became Prajāpati (lord of generation). And that Person which became Prajāpati is this very Agni (fire-altar), who is now (to be) built. He verily is composed of seven persons, for this Person (Agni) is composed of seven persons, to wit, the body (trunk) of four, and the wings and tail of three; for the body of that (first) Person (was composed of) four, and the wings and tail of three. And inasmuch as he makes the body larger by one person, by that force the body raises the wings and tail."[13].

It is about the bird-shaped altar (fig. 2) which will be extended, one area unit after the other, in accordance with techniques we will discover later. This energy of 7.5 p^2, a measure that expresses totality, must be embodied in concrete altar shape, depending on the wishes of whoever offers the sacrifice. It can be an isosceles triangle, a rhomb, or the form of a chariot wheel, for whoever wants to get rid of his enemies, a circle for whoever wants a village, the form of a rectangular trough for whoever wants food, the form of a bird for whoever wants heaven, and so on[14]. Concerning the method, these forms aren't just drawn, but generated from the basic figure, the p^2 square, stemming directly from the sacrificer's height, or from the 7.5 p^2 square; here is the basic reason for the Vedic geometrical research into the *transformation* of one form into another. That is no other than the transcription in geometrical terms of the abstract energy (the 1 or 7.5 p^2 square) which

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12 Extraordinary comparison, where the non-being is not the pure negative, the bare nothingness, but the being when it is not yet being: the non-being is in fact the potential being. Consequently, it is prophet, because it forecasts what is not yet. The primitive thought doesn't understand a thought separated from the action; thus the prophet, who forecasts and tells, is, by the same token, the demiurge who begets. I cannot resist the pleasure of quoting the text itself, which tackles with panache the mystery of the being coming from the non-being:

"Verily, in the beginning there was here the non-existent. As to this they say, 'What was that non-existent?' The *Ri*shis, assuredly,--it is they that were the non-existent. As to this they say, 'Who were those *Ri*shis?' The *Ri*shis, doubtless, were the vital airs: inasmuch as before (the existence of) this universe, they, desiring it, wore themselves out with toil and austerity, therefore (they are called) *Ri*shis." *Śatapatha Brāhmaṇa* 10-1-1. Translation Julius Eggeling, 1894. h

13 *Śatapatha Brāhmaṇa* 10-1-1. Translation Julius Eggeling, 1894. h

14 When one is immersed in the beautiful theory of the sacrifice, he can be disappointed by this vulgar exchange with the " gods ": I regale you, but, in compensation, you settle my business. And yet, there is no contradiction: " The selfish aims he pursues can't be matched against the public good, as long as the aims are those his place in society authorises " (Biardeau and Malamoud, *Le sacrifice dans l'Inde ancienne*, PUF 1976, quoted by Delire 2002, p. 11)

substantivizes itself in various beings (the altars) and animates them. Here are some examples:

The bird-shaped altar (fig. 2) for whoever wants to attain heaven is made of a body of 4 p^2, of two wings and a tail of 1 p^2 each; each wing, to make it stronger, is

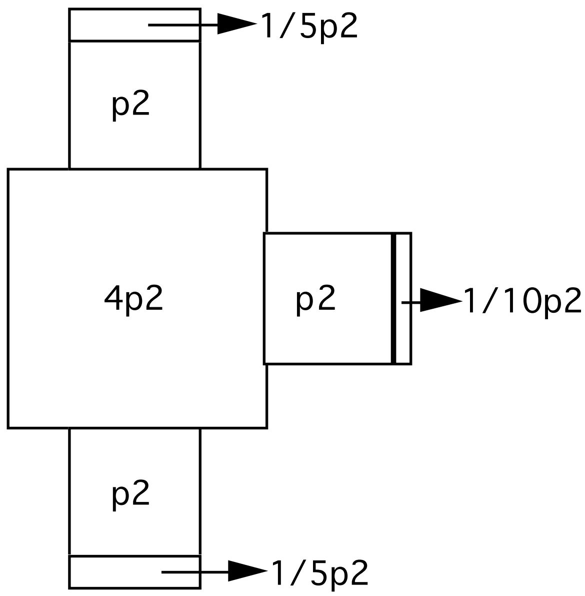

Figure 2. Bird-shaped altar

made one *aratni* longer, i.e. 1/5 *puruṣa*, and the tail one *pradeśa* longer, i.e. 1/10 *puruṣa*[15]. Altogether that comes to 4 + 3 + 2/5 + 1/10 = 7.5 p^2. The construction of the bird-shaped altar requires no other know-how than the building of squares and rectangles; and that is so important, that the *śulbasutras* give several methods, among which the one using Pythagorean triples. They are “ fundamentals ” which will be explained later.

“ Those who have many enemies should construct a *triangle* (fig. 3); this is the tradition ”[16]. To do that, a square is constructed with an area twice the area of the bird-shaped altar, then the middle of one side is joined with the ends of the opposite side[17]. In this way, an isosceles triangle with an area of 7.5 p^2 is obtained. This time, the technique is much less immediate. First, one has to know how to construct a square with an area of 7.5 p^2, equivalent to the sum of squares and rectangles

15 B. Śulb. 8-10.

16 Ā. Śulb. 12-4. The triangle is of course a triangle-shaped altar.

17 *Id.*, 12-5.

making up the bird-shaped altar. Then one has to know how to double this square: other “ fundamentals ” to be examined later.

“ According to tradition, those who wish to destroy existing and future enemies, should construct a fire altar *in the form of a rhombus* ”[18]. To do that, a rectangle is constructed placing side by side two 7.5 p^2 squares, and the middle of the sides east, west, north, south are joined (adopting the terminology of the text, fig. 3). In this way, a rhomb with an area of 7.5 p^2 is obtained.

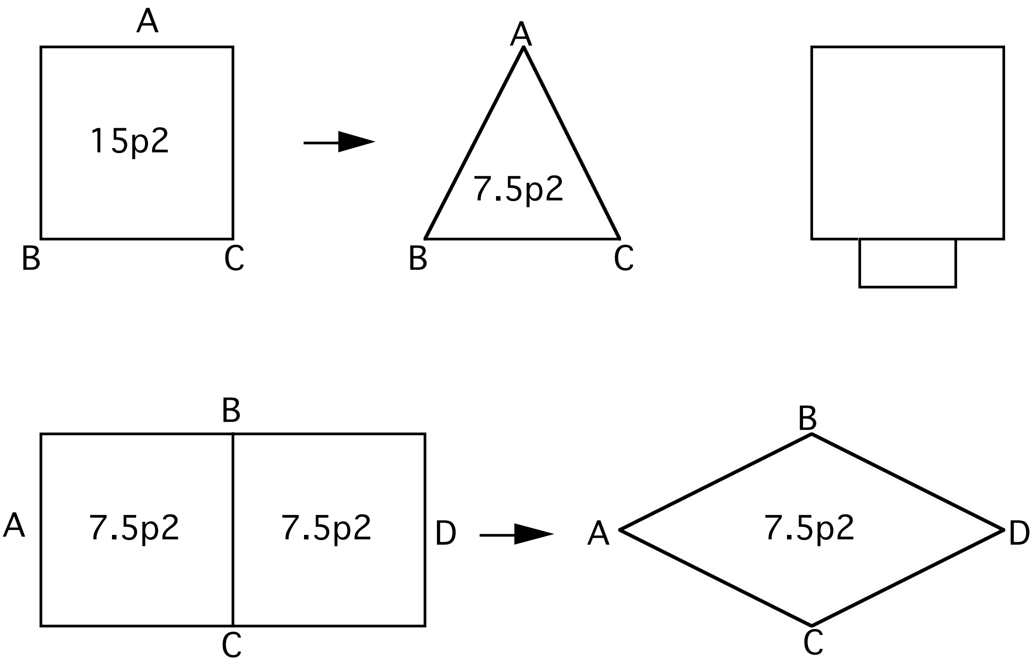

Figure 3. Altars with an area of 7.5 *square puruṣas*.
Triangle, trough and rhomb

According to *K. Śulb.*, 4-2, the *trough-shaped altar* (for whoever wants food, figure 3) is a square endowed with an appendix, an other square of which the area is the tenth of the total 7.5 p^2 area. The technique is as follows: cross-rule the 7.5 p^2 square in ten lines and ten columns, combine ten of the hundred obtained squares in one square and construct an other square with the remaining ninety ones. So they are two rectangles to be transformed into squares.

With the *chariot wheel-shaped altar*, we face the dreadful “ circling ” of the square, i.e. building a circle with an area equal to the area of a 7.5 p^2 square. In *B. Śulb.*, two types of wheels are given. The first is obtained easily (if I may say so) by constructing a disc with an area equal to the area of the 7.5 p^2 square, thanks to a rule explained in the beginning of the treatise; one “ fundamental ” more, and not

18 *Id.*, 12-7. A rhomb is defined by the construction in figure 3.

the least. As if the things were not long and difficult enough, Baudhāyana gives another way to construct the wheel, this time with the hub and the spokes (fig. 4). Hereafter, the detail:

1. Bricks are made with an area of 1/30 p^2 “ for measuring purpose ”.

2. 225 of these bricks are 225 x 1/30 = 7.5 p^2, the desired area.

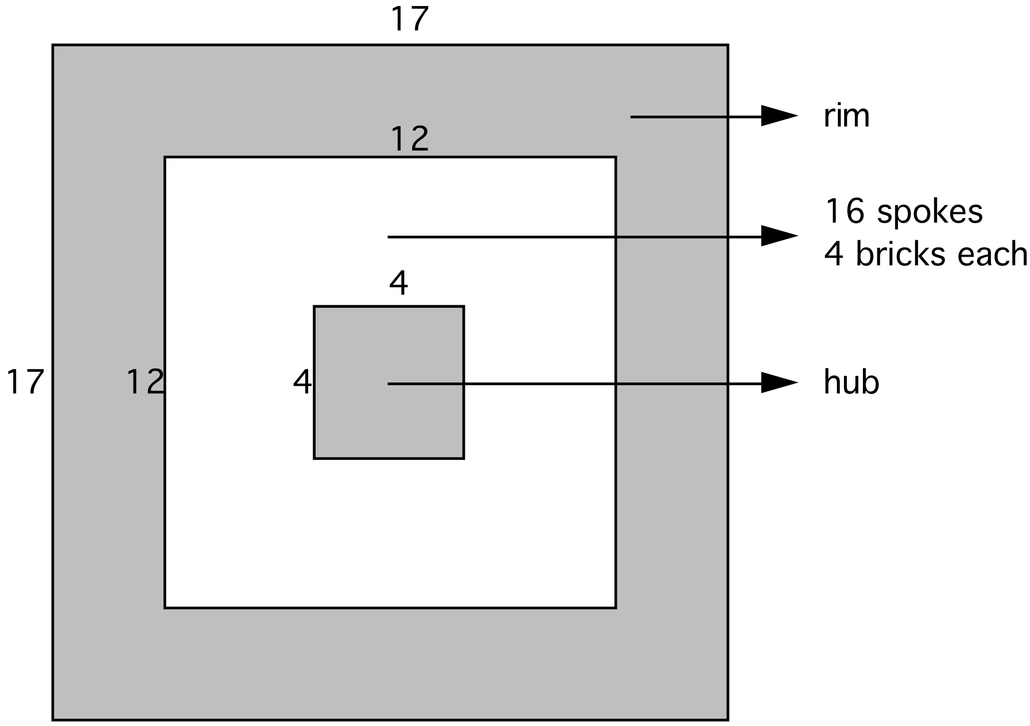

Figure 4. Arrangement of the bricks preceding the building
of the chariot wheel-shaped altar, with hub and spokes

3. With 64 bricks more, i.e. 289, a square of 17 x 17 is constructed; it will be noticed that 225 + 64 = 289, which can also be written $15^2 + 8^2 = 17^2$, takes into account the Pythagorean triple (15, 8, 17) well known in Vedic times. Instead of doing 17 lines of 17 bricks each, Baudhanya aligns 16 lines of 16 bricks each and places the remaining 33 (269 - 16 x16 = 33) on the sides: 16 on a side, 16 on the other, and the last in the “ corner ”, illustration of $(16 + 1)^2 = 16^2 + 2 \times 16 + 1^2$. Does this strangeness come from the mythical importance of the number 16? Let us remember that the earth can be associated with four, by the four cardinal points, and each part is fourfold as it contains also the whole, according to a “theorem” of the Vedic numerology: hence the role played by 16. The *Śatapatha Brāhmaṇa* reads for example: “These, then, are sixteen offerings, for man consists of sixteen parts, and the sacrifice is the Man (Purusha): hence there are sixteen offerings.”[19].

19 *Śatapatha Brāhmaṇa 11-1-6.* Translation Julius Eggeling, 1894.

4. From this square assembly of 289 bricks, 16 central bricks measure the hub area, 64 the area of the spokes and 64 the spaces between the spokes, that is 144 = 12 x 12 bricks. The remaining 289 - 144 = 145 are the rim. The figure looks now like fitting-together squares, which have only to be converted into circles: that is done, beginning with the hub, then with the internal and the external edge of the rim. The intermediate space between hub and rim, from which the 64 bricks of the spaces between the spokes have been removed (289 - 64 = 225 bricks = 7.5 p^2) is divided into 32 equal parts, of which 16 will be the spokes of 4 bricks each.

Extensions

The most spectacular construction is the one consisting in extending the bird-shaped altar of 7.5 p^2 area, so important as we know: " The fire-altar is that which is constructed in the likeness of the birds, that is, after the shadow cast by them when flying"[20]. Its area will have to be enlarged, unit after unit, up to 101.5 p^2. The dimensions are considerable; taking about 2.30 m. for the *puruṣa*, 7.5 p^2 is the area of a square of 6.30 m. in side, and 101.5 p^2 the area of a square of about 23.1 m in side. The necessity of this extension is clear: it is the extension of the vital energy. But why unit after unit, instead of doing it in one time, and why 101? The *Ṛg Veda* says:

> "The sacrifice drawn out with threads on every side, stretched by a hundred sacred ministers and one,
> This do these Fathers weave who hitherward are come: they sit beside the warp and cry, Weave forth, weave back."[21]

That shows that 101 "ministers" are necessary to " stretch " the sacrifice; this does not fit exactly with the extension unit after unit from 7.5 up to 101.5. But the instructions could have changed since the very ancient *Ṛg*. The *Śatapatha Brāhmaṇa* alludes several times to the number 101 and to the transition from 7 to 101, giving itself over to some numerical fiddling; its true origin is maybe in the belief that

> "That same (immortal light), indeed, is to be obtained either by the one hundred and one-fold (altar), or by a life of a hundred years: whosoever builds a one hundred and one-fold (altar), or whosoever lives a hundred years, he, indeed, obtains that immortality"[22]

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20 *B. Śulb.*, 8-5.

21 *Ṛg Veda* 10-130. Translation Ralph Griffith, 1896.

22 *Śatapatha Brāhmaṇa* 10-2-6. Translation Julius Eggeling (1897). h

and the additional "recapitulative" unit, frequent in the Vedic numerology[23], gives the looked for 101. Thus, the extension up to 101 would be the sign of the transition from Prajāpati to immortality.

The method, remarkable and very simple, in fact, consists in *changing the length unit and therefore, the area unit*. Thus, at every step, a new *puruṣa* will be constructed, and then the bird equal to 7.5 *new square puruṣas*. Here's how: we call E what the authors call the additional area over the basic altar of 7.5 p^2; as the altar has to extend from 7.5 p^2 up to 101.5 p^2, E varies from 1 up to 94. We have to construct the *new puruṣa* q thus:

$$7.5\, q^2 = (7.5 + E)\, p^2$$

So, we will have:

$$q^2 = (1 + E/7.5)\, p^2$$

We have to construct a square with an area of $(1 + E/7.5)\, p^2$, then with the side of this square taken as unit, construct the bird (fig. 2) with its body ($4q^2$), its wings and its tail ($3.5q^2$). Katyāyana[24], who seems to be the best mathematician of the four authors, gives three ways to go about this. To simplify matters, we'll take E = 1.

1st procedure (fig. 5): Construct a square C_1 with an area of 7.5 p^2 and add to it a square with an area of 1 p^2 to obtain a square C_2 with an area of 8.5 p^2. To construct the unit q. such as $7.5\, q^2 = 8.5\, p^2$, as $1/7.5 = 2/15$, we divide C_2 into 15 equal rectangular parts and two of them will be transformed into a square C_3 which is the " new " *square puruṣa*; the side of C_3 is the " new " *puruṣa*.

23 Example from *Śatapatha Brāhmaṇa* (10-2-6) : The one hundred and one-fold Pragâpati, doubtless, is the year, and thereto belong days and nights, half-months, months, and seasons. The days and nights of a month are sixty, and in the month, doubtless, the days and nights of the year are obtained; and there are twenty-four half-months, thirteen months, and three seasons (of four months)--that makes a hundred parts, *and the year itself is the one hundred and first part.*

By the seasons it is sevenfold,--six seasons (of two months), and the year itself as the seventh part. And he who shines yonder is the light of that year: his rays are a hundredfold, *and the (sun's) disk itself is the one hundred and first part.*

By the regions it is sevenfold,--the rays which are in the eastern region are one part, and those in the southern are one, and those in the western are one, and those in the northern are one, and those in the upper (region) are one, and those in the lower (region) are one, *and the disk itself is the seventh part.*

24 *K. Śulb.*, 5

Implied know-how: turn two squares into only one and turn a rectangle into a square.

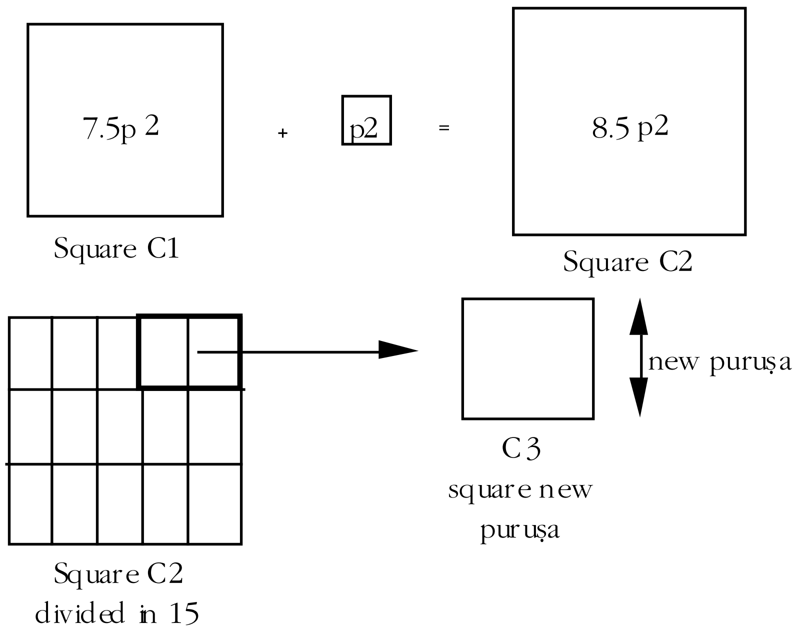


Figure 5. First Katyāyana's procedure for constructing a new *puruṣa* such as 7.5 *square new puruṣas* = 8.5 *square purusas*

2nd procedure: divide the square with a side of 1 *puruṣa* by 5 lines on both sides, turn five of these parts into one square, subtract 1/3 of this square and add the remaining to the initial square of 1 *puruṣa* side.

Translation: 1 + 2/3(5/25) = 1 + 2/15 = 1 + 1/7.5.

Implied know-how on top of the previous ones: construct a square equal to the third of a given square, subtract two squares to construct one only.

3rd procedure: divide the square with a side of 1 *puruṣa* by 7 (lines) on both sides, turn seven of these parts into one square, subtract a rectangle the sides of which are 1 *puruṣa* and 1 + 1/7 *aṅgula*; what supposes the turning of this latter into a square; add the remaining to the first 1 *puruṣa* square.

Translation: 7/49 - (1 + 1/7)1/120 = 1/7 - 8/7(1/120) = 2/15 = 1/7.5.

Adding that to the 1 *puruṣa* square, we obtain a square with an area of 1+1/7.5 p^2 the side of which is the looked for "new" *puruṣa*.

This method can be applied without difficulty to the cases where E ≠ 1. Both Baudhāyana and Āpastamba say more concisely: the excess E has to be divided in 15 parts and two of them added to the 1 p^2 square to obtain the new square unit. In actual language: 1 + (2/15)E is the new square unit ; 7.5 of such square give an area of: 7.5(1 + (2/15)E) = 7.5 + E.

The " fundamentals " of Vedic geometry

Orientation

First, the altars must be oriented. The first task of the Vedic geometer is to draw the east-west line, called *prācī*, then the north-south line. Katyāyana goes about in this way (fig. 6): a stake is planted and a circle traced with a cord with the stake for centre. Two stakes are planted in the two places where the shadow of the central stake touches the circle during the day: it is the east-west line. The cord is doubled, its ends tied to the stakes of the east-west line, then it is stretched by its middle on one side of the east-west line to obtain for example the point S; the point N is obtained in the same way, on the other side of the east-west line. From a geometrical point of view, we have here a very simple construction of a perpendicular NS to any straight line EW. We have already suggested[25] that it is a very ancient method, used for constructing right angles as well as for making sure of the horizontality of raised blocks.

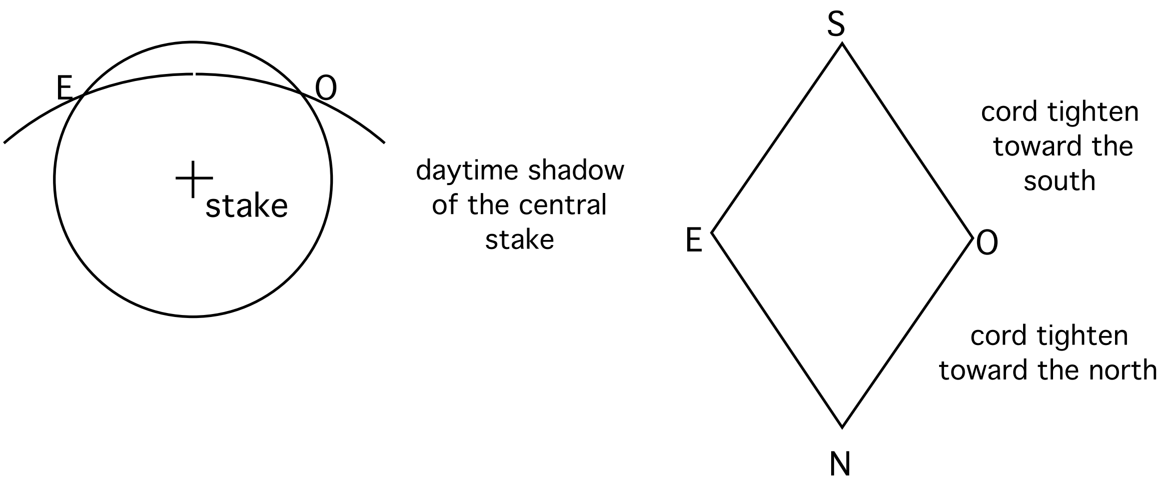


Figure 6. Determination of the line east-west (*prācī*) and of the line north-south

We will now give details about the various techniques given in the previous paragraph: construction of squares, property of the diagonal of the rectangle and constructions of figures with equivalent areas.

[25] See (Keller 1986) Chapter 3.

Construction of a square

The treatises give two principal kinds of methods to construct squares: either with stakes and a cord as long as the side, or with a cord marked in certain places to use Pythagorean triples. The marking of the cords, as well as the division of a segment into equal parts isn't considered in the Vedic texts to be a geometrical problem, as it needs no explanation. The observation of actual Vedic worshippers shows that the division was carried out by folding up the cord as many times as necessary.

The first construction of a square given by Baudhāyana[26] (fig. 7), starts from the segment EW oriented in accordance with the line *prācī*.

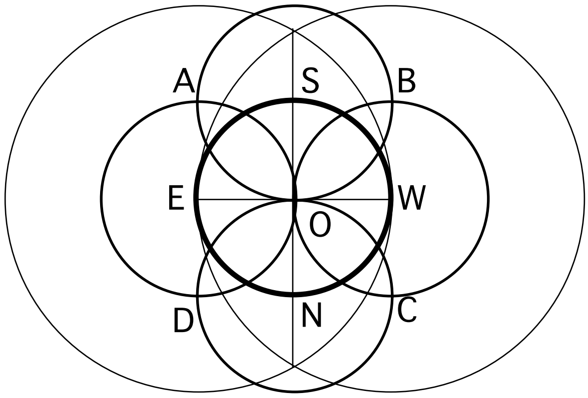


Figure 7. Construction of the oriented square ABCD from the segment EW.

A cord as long as EW is marked in its middle; its two ends are fixed to a stake in the middle O of the segment EW to trace the circle C_1 with O as centre and EW/2 as radius. The cord is then unfolded and one of its ends fixed in E to trace the circle with E as centre and EW as radius. On the same manner, a circle with W as centre and EW as radius is drawn. The straight line determined by the intersection of these two circles cuts the circle C_1 in N and S, north-south line. Then, 4 circles are traced with respectively E, N, W, S as centres and EW/2 as radius. They cut each other in O, and in A, B, C, D, the four corners of the square which had to be constructed.

Let us notice that the useful circles having for radius EW and EW/2, the construction can be done with a bamboo pierced at the ends and the middle, as shown by Āpastamba. On the other hand, the four circles with radius EW/2 form a rosette, a figure well known as an ornamental pattern in the region.

The second construction of a square makes use of various Pythagorean triples to trace right angles. Let us remember that a triple (a; b; c) is said to be Pythagorean if a, b, c are whole numbers such that $a^2 + b^2 = c^2$. As we know, the theorem usually

26 *B. Śulb*, 1-4.

known as Pythagoras' theorem is double: if a right-angled triangle has the sides of the right angle equal to a and b, and the hypotenuse equal to c, then $a^2 + b^2 = c^2$: reciprocally, if the sides a, b, c of a triangle verify the relation $a^2 + b^2 = c^2$, then the triangle is right-angled. The reciprocal is used for constructing squares with Pythagorean triples. As for tracing the line NS, the technique consists in stretching cords judiciously, on the following way (fig. 8). Let us construct the square with side a.

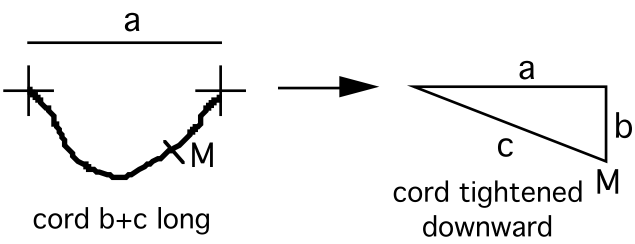
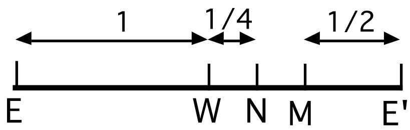
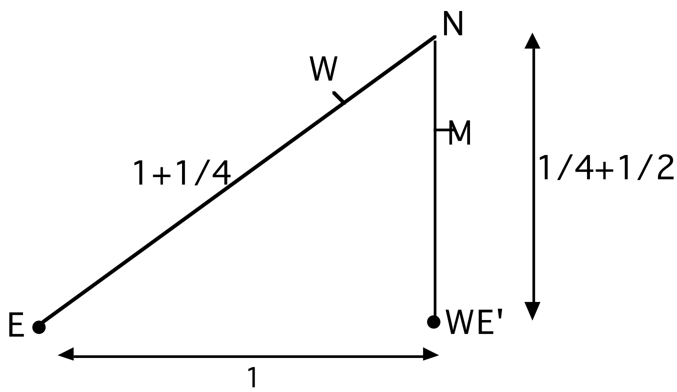


Figure 8. Stretching of the cord to obtain a right angle
(a, b, c) verify $a^2 + b^2 = c^2$

On the line EW two stakes are placed, distant by a one of the other. A cord with a length of c + b is made, with a mark dividing it into two segments of respective length c and b. After tying the cord to the two stakes distant by a, we seize the cord by this mark and stretch it on a side of the line EW; we obtain thus a triangle of sides a, b, c, and as $a^2 + b^2 = c^2$, this triangle is right-angled, with the side b oriented north-south.



Preparation of the cord EE' = 2EW = 2 *puruṣas*



Stretching of the cord seized in N, after the ends E and E' had been fixed to E, W, distant 1 *puruṣa*

Figure 9. Construction of the first corner M of a oriented square with a side of 1/2 *puruṣa*.

Here is the construction, according to Baudhāyana[27]. Let EE' be a cord (oriented east-west) 2 *puruṣas* long, W its middle (fig. 9). Let N be a mark placed at the quarter of WE' from W and M a mark placed at the middle of WE'. The two ends E E' of the cord are fixed on stakes placed in E and W and the cord is seized in N and stretched toward the south (upwards on our figure).

M is the "west corner" of the square in construction. Indeed, expressed in quarters of *puruṣa*, EW = 4, WN = 3 and EN = 5, and as $3^2 + 4^2 = 5^2$, the triangle ENW is definitely right-angled in W (which Baudhāyana doesn't specify). M is in reality the south-west corner of the square. Baudhāyana stops here, because it is easy to construct the other corners, stretching the cord toward the north (downwards in our figure), M will give the north-west corner, then by inverting the two ends of the cord, the same operations will give the south-east and the north-east corners. The four M points obtained by these four operations are the four corners of a square of 1 *puruṣa* side. A little further on, the same author gives another method, based on the triple (5, 12, 13).

Theorem of the diagonal of the rectangle

We will use the expression "theorem of the diagonal" because the Vedic authors expound it as a property of the diagonal of the rectangle: the square produced by the diagonal of a rectangle is equal to the sum of the squares produced by the two sides of the rectangle. The reverse is used, as we just have seen, but never expounded.

In mentioning the theorem of the diagonal only after having shown its use in the construction of a square, we follow the same order as in the *śulbasūtras*. After having explained the stretching of the cords to construct a square, Baudhāyana goes on[28]:

a) " The diagonal of a square produces the double of the area ". That means that a square of which the side is the diagonal of a given square has an area equal to the double of this square (fig 10).

b) " The breadth of a rectangle being the side of a given square and the length the side of a square twice as large, the diagonal equals the side of a square thrice as large.". Given a square, if a rectangle is constructed with the side and the diagonal of this square, the diagonal of this rectangle will "produce" a square with an area the triple of the area of the initial square. In actual terms, if c is the side of the initial

27 *B. Śulb*;, 1-5

28 *B Śulb.*, 1-9 to 1-13.

square, $c\sqrt{2}$ is the side of the double square, and the diagonal of the rectangle with sides c and $c\sqrt{2}$ is $c\sqrt{3}$. So, this diagonal “produces” a square with an area of $3c^2$.

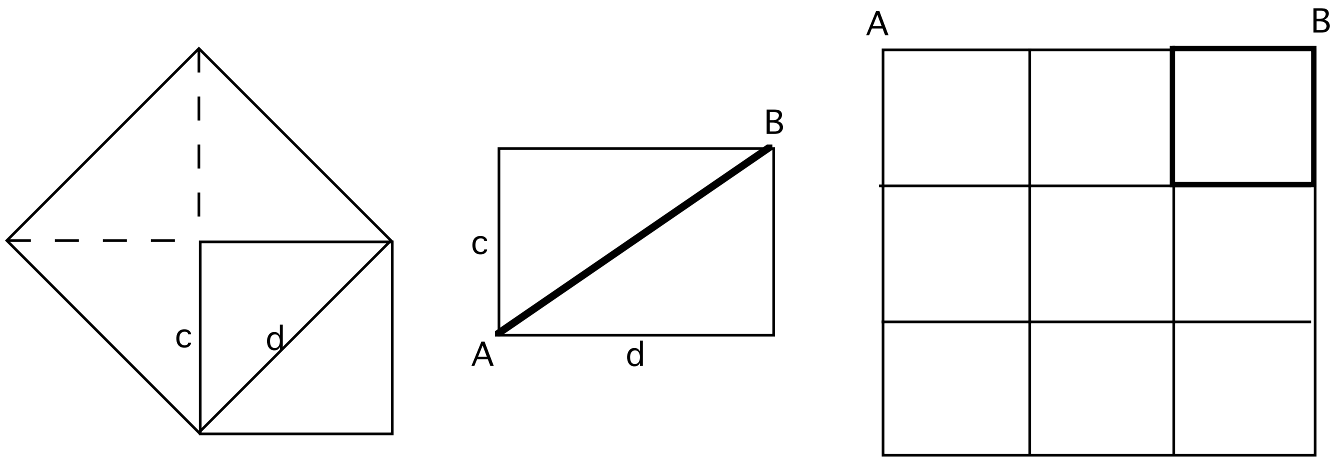


Figure 10.

On the left, the diagonal d of the square c “produces” the double of this square’s area. On the centre, AB is obtained as the diagonal of a rectangle of c and d sides. Hence, it produces an area of $3c^2$. The square of side AB on the right , divided in 9 equal parts will give a square with an area of $c^2/3$.

c) “ Thereby is explained the side of a square one-third the area of given square. It is the side of a square one-ninth the area of the square”. In actual terms, to construct $c^2/3$ from c^2, the square $3c^2$ is constructed as shown above, then $3c^2/9$; to divide the square $3c^2$ in 9 parts, each side will be divided into three by folding the cord on itself, as we have said before. Such a construction is required by a ritual which it would be useless to detail here, and by Katyāyana second procedure (see p.14).

d) “ The areas produced separately by the length and the breadth of a rectangle together equal the area produced by the diagonal ”. It is the theorem of the diagonal proper, expounded here without further ado. “ This is observed in rectangles having sides 3 and 4, 12 and 5, 15 and 8, 7 and 24, 12 and 35, 15 and 36 ”. The translation of J.-M. Delire is more interesting: “There is understanding of these diagonals when their sides are 3 and 4, and so on ”. In other words, the diagonals can easily be understood (calculated?) in some simple cases which come down to the Pythagorean triples (3; 4; 5), (12; 5; 13), (15; 8; 17), (7; 24; 25), (12; 35; 37) and (15; 36; 39). The authors of the *Śulbasūtras* know perfectly well that these simple cases are relatively rare, since the difficulties begin from the square of side 1; its diagonal, of a length equal to $\sqrt{2}$ in actual terms, is estimated by all the authors at:

$$1 + 1/3 + 1/3x4 - 1/34x3x4$$

Only Katyāyana adds "this is approximate". Did he mean that, by refining the approximation (we will see later how to do that), the exact value would be obtained, or did he guess that it was possible to refine indefinitely and never obtain the exact value ? Nothing allows us to answer that question.

The reciprocal part of the theorem of the diagonal is never expounded in the *śulbasūtras,* although it is the one which "works" when the triples are used for constructing the right angle. Actually, if Baudhāyana, by stretching his cords, obtains effectively a right-angled triangle, it is because its sides are proportional to 3, 4, 5.

We should not give too much importance to the order of the previous propositions, as we might be tempted to, in the hope of reading into it an history of the discovery of the theorem of the diagonal. The others authors of the *śulbasūtras,* in fact, expound roughly the same results, but in a different order. For example, in Āpastamba, the order, more logical, is the following:

1. Theorem of the diagonal
2. The diagonal of the square produces two times the area
3. Construction of the triple square, and of a square equal to the third of a given square.

On the other hand, what is common to all these authors is that their texts begin with the construction of squares, purely geometrical constructions or based on Pythagorean triples, as we just have shown, and the wording of the theorem comes afterwards.

There is no way here to escape the question of the how. It's true that in our investigation of the fundamentals of the Vedic geometry, we came across no proof, in the contemporary sense, be it for the simple construction of the NS line perpendicular to the EW line, for the purely geometrical construction of a square, or for the theorem of the diagonal. But while the first two can be considered as visually obvious, it is certainly not the case for the theorem; what is more, even once the property of the diagonal has been discovered, that says nothing about the way to construct Pythagorean triples. We don't know, and we will probably never know, how the Indians have discovered the theorem, and the same can be said for the Babylonians in Hammurabi's times, for the Greeks of the classical era or for the Chinese of the Han's age. But we have been left some serious clues which allow us to form some ideas about its discovery.

Baudhāyana begins saying that " the diagonal produces the double of the area ", which can be immediately seen. From two equal squares, a single square is easily constructed. The fig. 11 shows how, using the same method of the substitution of equal areas, and without any calculation, in a diagram which could come from an adaptation of the first one, a single square of side h is made from two squares of sides a and b. h is the hypotenuse and a and b the right angle's sides; so $h^2 = a^2 + b^2$.

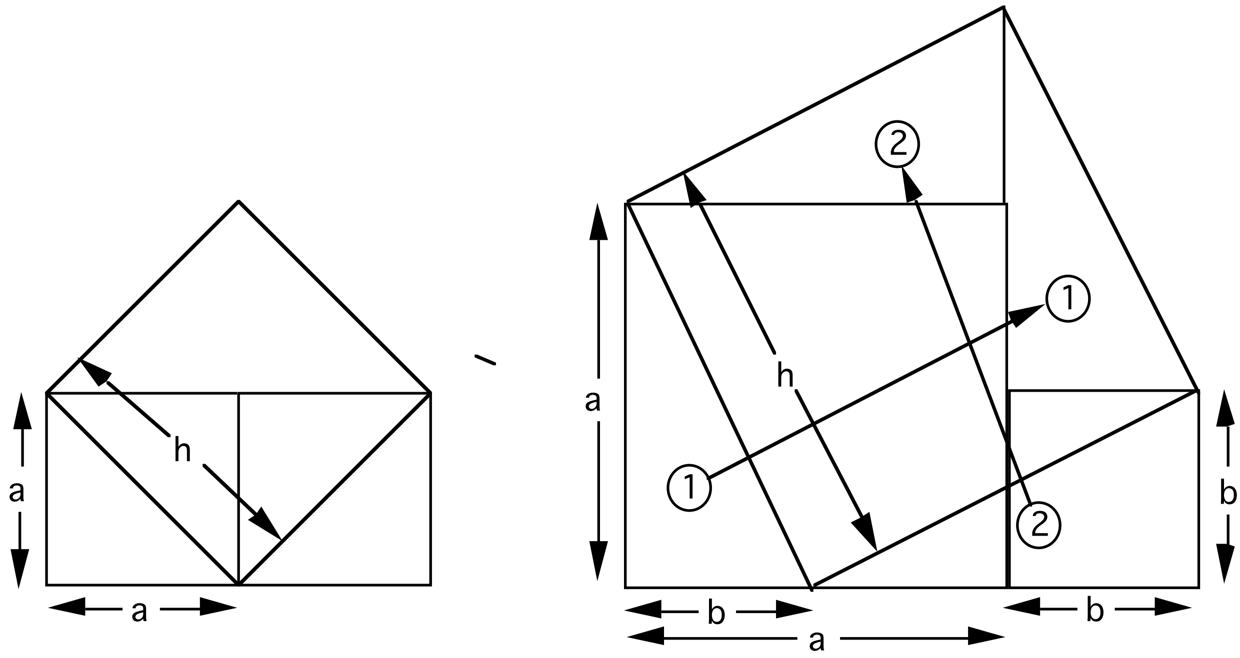


Figure 11. Visual demonstration of the theorem of the diagonal.
To the left, a special case; $h^2 = 2a^2$. To the right, general case; transformation of two squares of a and b sides into a square of h side. By cutting out from the figure formed by the two squares the two triangles 1 and 2 and by placing them as shown by the arrows, we see that $h^2 = a^2 + b^2$.

This very visual demonstration, given in an Indian treatise of the 16th century[29], is usually attributed to Thâbit ibn Qurra (9th century); the first explicit demonstration known in India is ascribed to Bhaşkara II (12th century), which can also be read on the same figure[30], but less obviously. Simple drawings allow us to "see" the theorem of the diagonal. It is not so with the reverse, that is to say that, if $a^2 + b^2 = h^2$, then a and b are the sides of a rectangle and h its diagonal. One is entitled to wonder if the reverse question has ever crossed the mind of our cord stretchers: has the triangle of sides 3, 4, and 5 ever tricked anyone ?

Whatever the case, a serious clue allows us to think that the triples, so handy to construct a right angle, were not discovered by chance or by trial and error. Let us

29 The Jyeşhţhadeva's *Yuktibhāşā.*

30 Figure present in a Bhaşkara I's text (7th century), but without any explicit reference to the theorem of the diagonal.

examine the method given (without proofs) by Katyāyana[31] to construct a square with an area equal to the one of n squares equal the one to the other. It is enough, says he, to construct an isosceles triangle with the base AC = n - 1 and the sides AB = BC = (n + 1)/2. Let BH be the height (fig. 12). As

$$BH^2 = AB^2 - AH^2 = ((n + 1)/2)^2 - ((n-1)/2)^2 = n$$

BH = √n is the side of the looked for square.

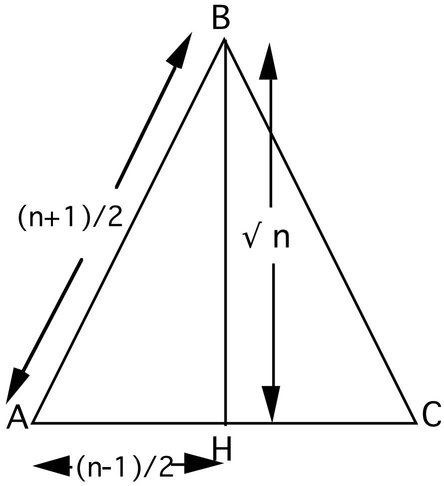


Figure 12

Now, if we take apart the right-angled triangle AHB, its sides are (n - 1)/2, √n, (n + 1)/2. If we take √n = p, the sides are respectively:

$$(p^2 - 1)/2 \; ; \; p \; ; \; (p^2 + 1)/2$$

formula which allows to make an infinity of Pythagorean triples: if p is an odd number successively equal to 3, 5, 7 for example, we obtain the triples (3; 4; 5), (5; 12, 13), (7; 24; 25). If p is even, the formula isn't suitable as it is: everything has to be multiplied by 2, to form the triples (p^2 - 1 ; 2p ; p^2 + 1) which is also Pythagorean; with p successively equal to 2, 4, 6 for example, we obtain (3; 4; 5), (8; 15; 17), (12; 35; 37). Thus, we have obtained all the triples given by Baudhāyana, except for the last, (15; 36; 39), proportional to (5; 12; 13). We can reasonably suppose that our authors have made the connection between the triangle AHB and the triples

$$((p^2 - 1)/2 \; ; \; p \; ; \; (p^2 + 1)/2) \text{ and } (p^2 - 1; \; 2p; \; p^2 + 1),$$

known in Greece long before Euclid according to Proclus[32]: he ascribes the first to Pythagoras, the second to Plato.

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31 *K. Śulb.*, 6-7.

32 Proclus, 1948.

Adding and subtracting two squares

Adding and subtracting two squares are two immediate application of the theorem of the diagonal, as shown in the figure 13. We have just shown how Katyāyana, with the same theorem, adds in one go n equal squares. Additions and subtractions take place in the different methods to increase the bird-shaped altar.

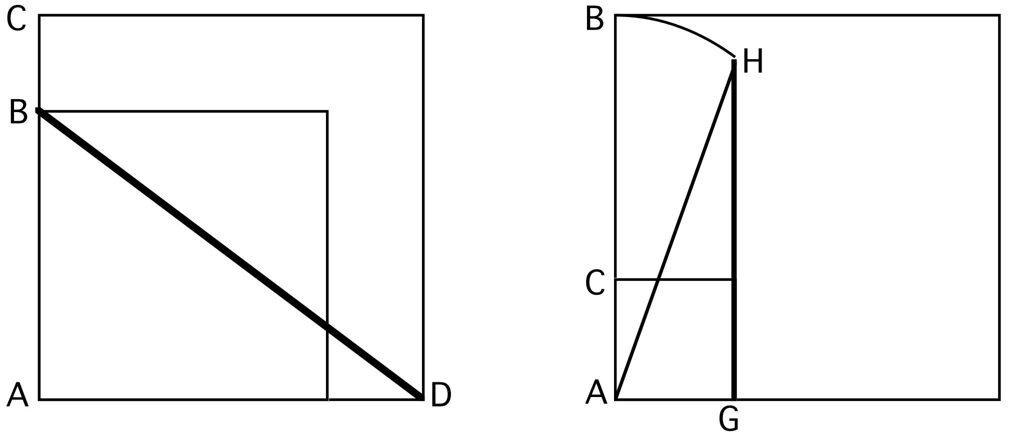
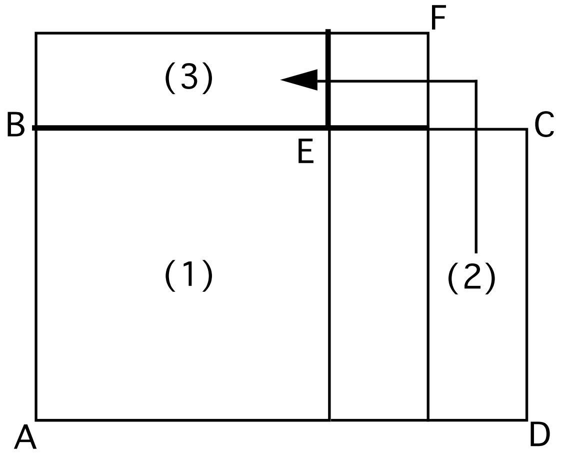


Figure 13. Applications of the theorem of the diagonal

On the left, BD is the side of the square equal to the sum of the squares AB and AC.
On the right, AB = AH, thus $GH^2 = AH^2 - AG^2 = AB^2 - AC^2$. Thus GH is the side of the square equal to the difference of the squares AB and AC.

Transforming a rectangle into a square with the same area, and vice versa

The first transformation plays a key role in the expansion of the bird-shaped altar, it is carried out in the most rigorous way: the rectangle is first transformed into a "gnomon", i.e. the difference of two squares and we know how to construct a square equal to this difference. The figure 14 shows the process.



Transformation of a rectangle into a square

Let a rectangle ABCD (with, for example, AD>AB) be transformed into a square. The square (1) of side AB is constructed in the rectangle. The remaining rectangle, with the diagonal ED is divided in two. The half (2) is placed in (3). Thus, the rectangle ABCD has been transformed into a figure which is the difference of the squares of the respective diagonals AF and EF, and we know how to construct this difference (fig. 13).

Figure 14

The methods for the second transformation are incomplete or incomprehensible. *B. Śulb.* 2-3 explains how to transform a square into a rectangle of which one of the two sides is the diagonal of the square; the process is correct. But in *B. Śulb.* 2-4, as in *Ā Śulb.* 3-1, the side of the rectangle to be constructed isn't specified, and the explanation is obscure.

The kind of cut-out at work in the transformation of a rectangle into a square gives us the most sensible explanation, in my opinion, of the way the Indians discovered the approximation

$$1 + 1/3 + 1/3x4 - 1/34x3x4$$

of the diagonal of the square of side 1. This explanation, due to B. Datta[33], gives in a very natural way the approximation, in the form of a sum of "unit" fractions as in the *śulbasūtras*. The problem is to construct (differently as by using the theorem of the diagonal) a square equivalent to two given equal squares (fig. 15). The second square is divided into three equal slices, of dimensions 1 and 1/3. Two of these slices are placed along two sides of the first square, which leaves a "corner" of 1/3 x 1/3; The last slice is divided into three squares 1/3 x 1/3, one of which is placed in the "corner", completing a square of (1 + 1/3) side. Each of the two last squares of the last slice is divided in four slices, which gives eight slices of 1/3 length and 1/4 breadth, which are arranged on two sides of the square $(1 + 1/3)^2$. The two starting squares are thus

been transformed in a square of (1 + 1/3 + 1/3 x 4) side by $1/(3 \times 4)^2$ excess.. This excess is split between the two sides of the square, i.e.:

$$\frac{\frac{1}{(3x4)^2}}{2\left(1 + \frac{1}{3} + \frac{1}{3x4}\right)} = \frac{1}{3x4x34}$$

Finally, a square is thus obtained of 1 + 1/3 + 1/(3 x 4) - 1/(3 x 4 x 34), but the process can be carried on.

33 Given in Sen and Bag 1983, p. 167.

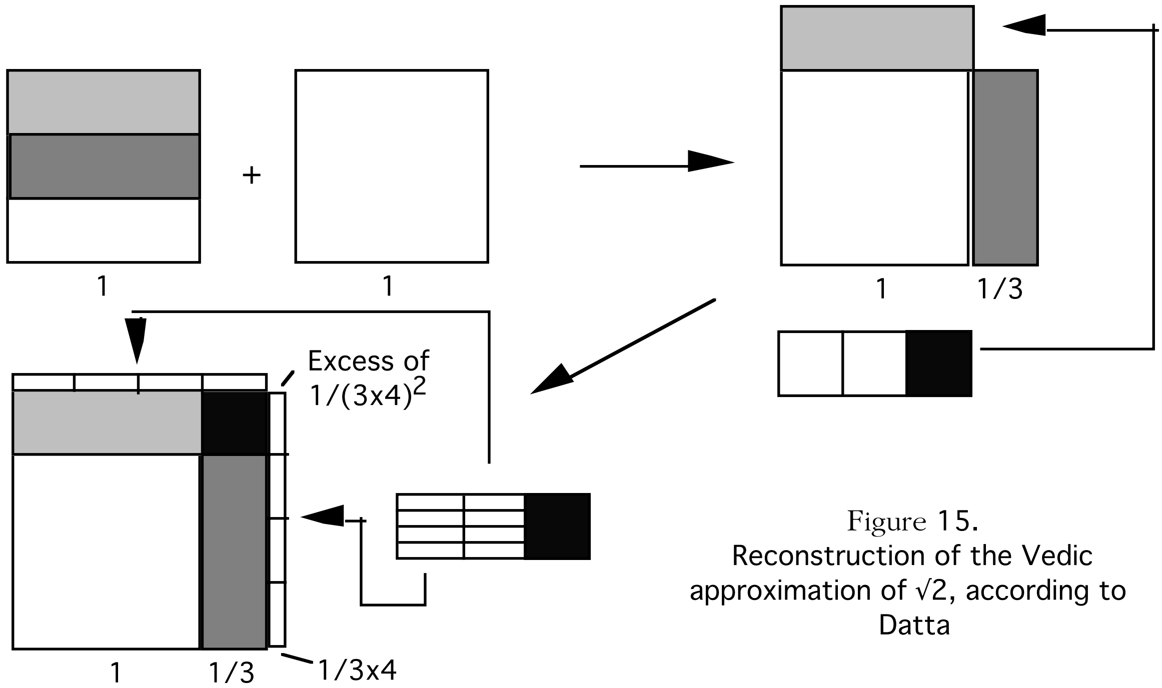


Figure 15.
Reconstruction of the Vedic approximation of √2, according to Datta

The last division of fractions can't be carried out without a process equivalent of our reduction to a common denominator. A unit's change could help: supposing the starting unit is the *puruṣa*, and knowing that a *puruṣa* is 120 *aṅgulas* and an *aṇgula* is 34 *tilas*[34], the last excess of $1/(3 \times 4)^2$ is equal to $120/(3 \times 4)^2$ = 100 square *aṅgulas* = 100 x 34 x 34 square *tilas*. Shared between the two sides of the (1 + 1/3 + 1/12)*puruṣa* = 170 *aṅgulas* = 170 x 34 *tilas* each, a breadth is obtained of (100x34x34)/(2x170x34) = 10 *tilas*, which converted in 10/34 *aṅgula*, then in 10/(34x120) *puruṣa*, gives the looked for result 1/(34x12).

Other methods could be invented, following Datta's idea, but slicing differently the second square. If we slice it, for example, in five equal rectangles[35], instead of three, we'll obtain rapidly the approximation 1+ 2/5 + 1/(14x5); or slicing it in two equal rectangles, we obtain 1 + 1/2 + 1/(4x3) - 1/(4x3x34)

The Vedic approximation is of excellent quality: the first five decimals are exact. On top of that, we note that the series

1 + 1/3; 1 + 1/3 + 1/12; 1 + 1/3 + 1/12 - 1/(12x34)

34 *B. Śulb.*, 1-2.

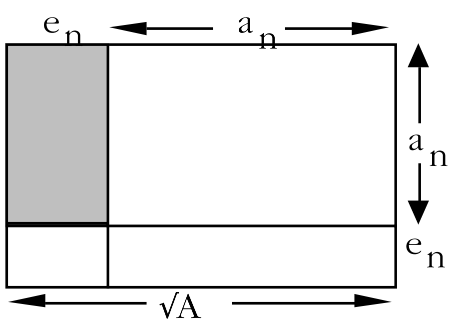
35 Delire 2002, p. 118.

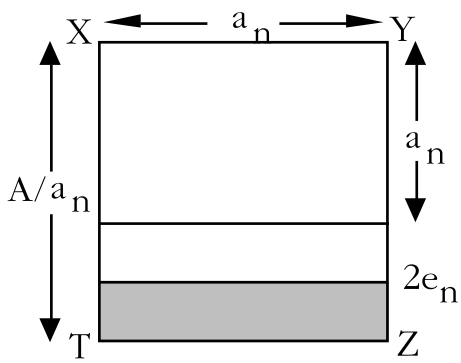
a_1	a_2	a_3	a_4
4/3 = 1 + 1/3 dividing the second square in three slices	17/12 = 1 + 1/3 + 1/12	577/408 = + 1/3 + 1/12 - 1/(12x34)	
7/5 = 1 + 2/5 dividing the second square in five slices	99/70 = 1 + 2/5 + 1/514		
1	3/2 = 1 + 1/2	17/12 = 1 + 1/2 + 1/12	577/408 = 1 + 1/2 + 1/12 - 1/(12x34)

corresponds to the series of the approximations of √2 obtained by Heron's method[36]:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

with $a_1 = 1 + 1/3$. By taking $a_1 = 1 + 2/5$ or $a_1 = 1$ we obtain the two other approximations suggested above, resulting from other ways of slicing the second square.





Let us estimate a square of √A side. Let an be a_n approximate value by default, and e_n the error commited.

The same final result is obtained if an is an approximate value by excess, with a diagramm slightly different.

The grey area of the left figure is placed as shown above and the lower bottom corner equal to e_n^2 is neglected.

We'll have: area XYZT = A = $a_n(a_n + 2e_n)$, hence:

$a_n + 2e_n = A/a_n$, and:

$a_n + e_n = 1/2(a_n + a_n + 2e_n) = 1/2(a_n + A/a_n)$.

As next approximation of $\sqrt{A} = a_n + e_n$, we'll thus take $a_{n+1} = 1/2(a_n + A/a_n)$

Figure 16. Calculation of a square root by the Heron algorithm

36 Heron of Alexandria, second century A.D. ?

There is nothing surprising about these coincidences, as the Heron method itself can be justified by the same type of handling as the one we have ascribed (according to Datta) to the authors of the *śulbasūtras* (fig. 16).

Transforming a square into a circle with the same area, and vice-versa

The first problem is the "circling" of the square, the second the squaring of the circle. To solve the first, Baudhāyana (*B. Śulb,*, 2-8) enjoins us to take the radius of the circle equal to half the side of the square plus a third of the difference between half its diagonal and half its side (fig. 17).

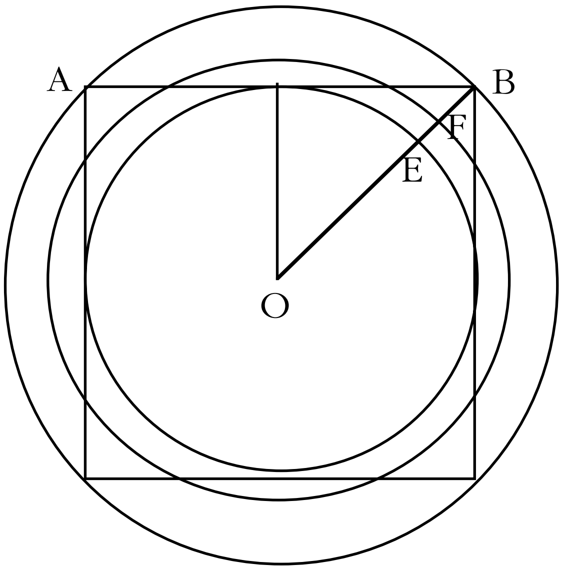


Figure 17. Transformation of the square of side AB into a circle with the same area.

The looked for circle has for radius OF = OE + EB/3, where OE is half the side of the square and EB the difference between half the diagonal and half the side.

The three other authors give the same method, and none of them mentions that it's only an approximation. Āpastamba adds: " This is the circle, for as much is added as is cut off "[37]. One can easily think that such a method is the result of empirical research: the circle having half of the diagonal for radius is too big, the one having half of the side, too little. The adopted solution would be a compromise satisfactory to the eye. If we take as radius half the side plus a third of the difference between half the diagonal and half the side, the areas of the circle jutting out of the squareare equivalent to the areas of the square jutting out of the circle. If we want to estimate the accuracy of this method from a contemporary point of view, we'll note that it leads to a ratio $d/s = (2 + \sqrt{2})/3$ (where d is the circle's diameter and s the side of the square) and that the corresponding approximation of p[38] is of 3.088312; the error amounts to about 1.69%. We can also note that the choice of the third of the difference gives a much better result than the fourth (p = 3.28) or the half (p = 2.74) .

37 *Ā. Śulb.*, 3-2.

38 The area of the circle is $\pi d^2/4 = s^2$, thus $\pi = 4(s/d)^2 = 4/(d/s)^2$.

The circling of the square is used ritually several times, for example in the building of the chariot wheel altar we examined above. On the other hand, the squaring of the circle is never used, so far as I know, and yet Baudhāyana gives two methods to carry it out. First method, given by Baudhāyna only: the side of a square having the same area as a given circle is the fraction

$$1 - \frac{1}{8} + \frac{1}{8x29} - \frac{1}{8x29}\left(\frac{1}{6} - \frac{1}{6x8}\right)$$

of the circle's diameter. A second method, given by all is: the side of the square is 13/15 of the diameter, and Baudhāyana adds: " this gives the approximate side of the square ", letting us think that the method he had given previously is exact. Nothing convincing has been found until now about the origin of these methods[39].

Such are the fundamentals of the geometry of the *śulbasūtras*. Probably serious interests or an unfailing Vedic faith were essential to implement such long and complex rites. And we have not mentioned the accompanying arrangement of the bricks on every layer of the altar, nor the interminable reciting going with every step! In the previous paragraphs, we have shown how the spirit of these labours of Hercules could be deduced from the theory of the sacrifice: now we have explored the letter of it, we'll show how it could be analysed as an embryonic corpus of geometry. But, before doing that, we'll look into the striking similarity between the problems we have just gone over and the problems dealt with in books I, II and VI of Euclid's *Elements*.

Analogies with some problems of Euclid's *Elements*.

Although the *Elements* don't offer any construction with the help of Pythagorean triples, and don't make any area or length calculation, although the *Elements*, in their deliberate and systematic hypothetico-deductive form, definitely reflect another mathematical world from that of the *śulbasūtras*, there is nevertheless a kind of common spirit between the two treatises.

First, they use the same elements of construction, the straight line and the circle, in the form of requests (postulates) by Euclid, with an ungraduated cord and stakes by the ritual practitioners. The *śulbasūtras* have a basic figure, the square, carefully

39 Except perhaps the latest paper on the subject: (Kichenassamy, 2006).

constructed on a given straight line, the east-west line, which is the pivot of all the considered constructions; the basic figure being a quadrilateral, it is normal that the Pythagorean theorem appears as property of the diagonal of the rectangle. The *Elements* have a basic figure, the triangle, of which the construction on a straight line is the object of the first proposition, and which is the pivot of every subsequent development; it is thus natural that the Pythagorean theorem appears as a property of the hypotenuse of a right-angled triangle.

We have seen that, to construct the north-south line from the east-west one, the Vedic practitioners stretched twice a cord by its middle, after having fixed its ends on two stakes placed on the east-west line; the process amounts to using two symmetrical isosceles triangles, then joining their vertices. Euclid also stretches the cord in his own way, systematically using the isosceles triangle (in reality the equilateral triangle, but only its isoscelean properties are used) to construct the perpendicular to a given straight line (Book I, prop. 11 and 12), although that is not the simplest way to carry it out. That might be due to the necessities of the general arrangement, but also, more simply, to the influence of the traditions set up by the Neolithic master-builders.

The parallel is even more striking in the nature of the posed problems: in the *śulbasūtras*, as in the *Elements*, Books I and II, the central problem is the construction of surfaces of equivalent areas[40]. The methods of the Vedic practitioners, that is to say, for the essential, the theorem of the diagonal and the transformation of a rectangle into the difference of two squares, are to be met in the *Elements*, Book II[41]. The reader will remember that, to transform a rectangle into a square, the rectangle is first transformed into a square figure from which a corner is missing, i.e. a difference of two squares, and this figure, in its turn, is transformed into a square thanks to the theorem of the diagonal (fig. 14). Euclid, by whom the square deprived of a corner is called "gnomon", performs this operation, but in a much more general perspective than in the *ṣulbasūtras*. His aim is actually the proposition 14, the last of the Book II, the climax of the whole, where a square equivalent to any rectilinear figure is constructed. Today, we would speak of the squaring of any rectilinear figure whatsoever. Here below, how Euclid carries it out:

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40 The expression is abusive, as, if the *śulbasūtras* deal really with areas, i.e. with measured figures, it's not the case with Euclid's treatise, where only equivalents fields are at stake, without any number being associated to it. This misuse of language is of no consequence in the setting of this study.

41 In what comes after, I use B. Vitrac's translation, *Euclide*, 1990.

Let, as in the proposition II-5, AB be a segment, C its middle and D any point of this segment (fig. 18); let (AH)[42] be the rectangle " contained by AD, DB " (rectangle on AD and DH = DB) and (CF) the " square on CB ". Euclid demonstrates that the rectangle (AH) plus the " square on CD " is equal to the square (CF), which means that the rectangle (AH) is equal to the difference between the square (CF) and the square on CD equal to (LG), therefore to the gnomon NOP. It is easily seen on the figure: that (AH) = (AL) + (CH); but (AL) = (CM) since C is the middle of AB, and (CH) = (HF). So, finally, (AH) = (CM) + (HF) = gnomon NOP.

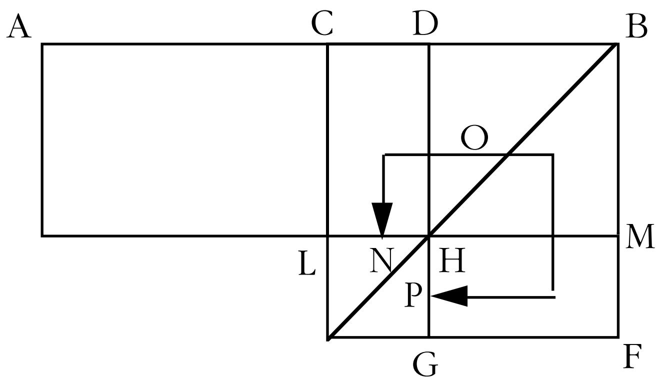

Figure 18. Euclid's II-5 proposition

The rectangle (AH) is equal in area to the gnomon (NOP)

If we try, with this proposition, to transform a rectangle such as (AH) in a difference of two squares, the process will slightly differ from the *śulbasūtras*: we add to the rectangle (AH) the square on DH to obtain the rectangle (AM); this rectangle is divided lengthways in two in C and a rectangle equal to (CH) is placed on the side HM. The gnomon NOP is equal to the initial rectangle (AH).

On the other hand, the proposition II-6 gives us exactly the method used in the *śulbasūtras*. Let AB be a segment, C its middle and D any point on the straight line AB, but not between A and B (fig. 19); let (AM) be the rectangle " contained by AD, DB " (rectangle on AD and DM = BD) and (CF) the " square on CD " Euclid demonstrates that the rectangle (AM) plus the " square on CB " (equal to (LG)) is equal to the square (CF), what means that the rectangle (AM) is equal to the gnomon NOP, difference between the squares (CF) and (LG). Actually, (AM) = (AL) + (CH) + (BM); as (AL) = (CH) = (HF), we have (AM) = (HF) + (CH) + (BM) = gnomon NOP.

42 In what comes after, (XY) will refer to the rectangle of diagonal XY

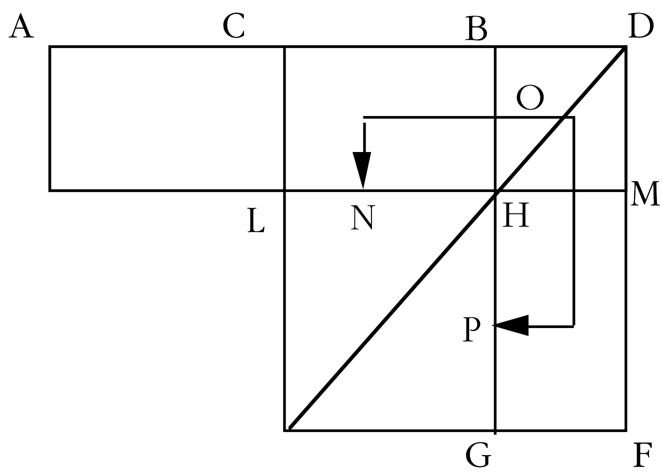

Figure 19. Euclid's II-6 proposition
The rectangle (AH) is equal n area to the gnomon (NOP)

Resulting from this proposition, the construction of a gnomon equal to a given rectangle (AM) would be as follows: take away from AD the breadth DM of the rectangle and divide AB in two at the point C; carry the rectangle to (HF), which will give the gnomon NOP. That is exactly the construction of the *śulbasūtras*.

Book II ends with the proposition 14 that leads to the squaring of any rectilinear figure. Euclid transforms first the given rectilinear figure into a rectangle (BD), thanks to the propositions I-44 and I-45, a method called the application of areas,

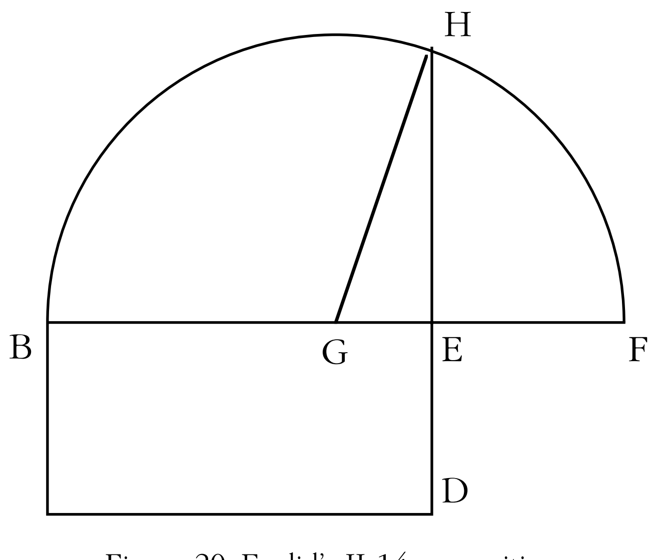

Figure 20. Euclid's II-14 proposition
Construction of EH, side of the square equivalent to the rectangle (BD)

apparently not known to the Vedic practitioners; but they don't need it, since they start from a rectangle and not from any rectilinear figure. Euclid then constructs BF = BE + ED, and divides BF into two in the point G (fig. 20); thanks to II-5, he knows that (BD) is equal to the gnomon $GF^2 - EG^2$. To finish, it is sufficient to construct, thanks to the circle of centre G and radius GF, a right-angled rectangle with a side equal to EG and the hypotenuse to GF; The second side of the rectangle verifies: $EH^2 = GH^2 - GE^2 = GF^2 - GE^2$, that is $EH^2 = (BD)$, and the problem is solved.

So we have the sequence: transformation of a rectangle into a gnomon, followed by the application of the theorem of the diagonal, as in the *śulbasūtras*.

To finish with the gnomon, we can note that the proposition II-8 of the *Elements* proves what Katyāyana maintains without proof. We remember that he gives a very quick method to transform n equal squares (of side 1, for example) into only one: to construct an isosceles triangle with a base of n-1 and two sides equal to (n + 1)/2; the height h relating to the base gives the side of the looked for square (fig. 12). We have actually,

$$h^2 = \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2 = n$$

the square of side h is equivalent to n square of side 1. To demontrate in the Euclidian way that:

$$\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2 = n$$

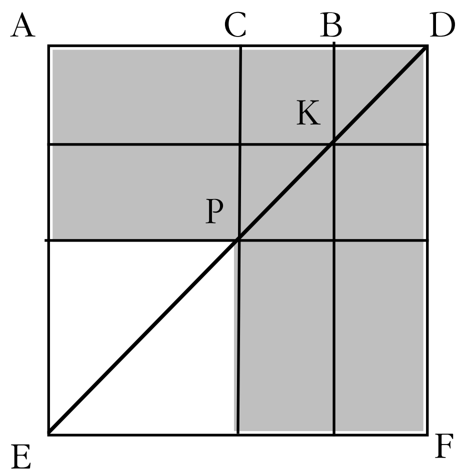

Figure 21. Euclid's II-8 proposition

let us take a segment AB, a point C on it and extend AB by BD = BC (fig. 21). The proposition II-8, in modern language, maintains that $4AB \times BC + AC^2 = AD^2$. Putting AB = n and CB = BD = 1, it is perfectly obvious on the figure that the double grey gnomon is equal to four times the rectangle (AK) of which the area is n. As this double gnomon is also the difference between the squares (AF) and (EP) of n - 1 and n + 1 sides, we obtain in modern language:

$$4n = (n+1)^2 - (n-1)^2 \text{ from where : } n = \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2$$

There is another striking parallel to be made between Book VI of the *Elements*, dealing with the construction of homothetical figures on the one hand, and the famous section of the *śulbasūtras* about the extension of the bird-shaped altar on the other. But if the nature of the problem is the same, although reduced to a specific figure in the *śulbasūtras*, the theoretical gap with the Euclidian treatise is even more considerable than in the case of the use of the gnomon. The difference lies not only in the fact that Euclid subjects everything to proof, nor in the fact that his central proposition in this domain: " VI-25: to construct a same figure similar to a given rectilinear figure and equal to another given rectilinear figure " is much more general; the theoretical leap forward is to have constructed similarities, whatever their ratio, rational or not. And that is the reason why a lengthy preparation was required, with the brillant Book V which introduces the theory of the ratios of the magnitudes. In the *śulbasūtras*, on the other hand, the technique outlined for the bird-shaped altar could admittedly be extended to any rectilinear figure, but on condition that the ratios of the areas be rational.

Despite the theoretical gap, the connection between Book VI and the extension of the altar, strikes me as being very significant. In both cases, the matter is about methods of great importance, and seen as such. That this is so for the *ṣulbasūtras*, we have already sufficiently emphasized. As for the proposition VI-25, it was considered by the Greek commentators as the masterpiece of the *Elements*. On that subject, Plutarch said:

> " Here we have actually one of the essential theorems, or better problems, of geometry: two figures being given, make a third equal to the one and similar to the other, a discovery about which, according to tradition, Pythagoras was said to have offered a sacrifice. That is certainly more elegant and more clever than the famous theorem which demonstrated that the hypotenuse is equal in power to the sides of the right angle "[43].

Proposition VI-25 would perhaps have been be the culmination of very ancient research, going back in time to Pythagoras, and giving an even more prestigious result than the theorem of the hypotenuse. Proclus, in his *Commentary of the Elements*, refers to the problem of the application of the areas, " former discoveries of the Muse of the Pythagoreans ", and he gives an example with *measured* areas,

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43 Quoted by M. Caveing, 1997, p. 329. The fact that Plutarch quotes the proposition in geometrically inadequate terms is here of no importance.

exactly as a Vedic practitioner could have done[44]. Holding plausible what Proclus and Plutarch report, in spite of the distance of several centuries that separates them from Euclid, and comparing it with the Vedic ritual, the following reconstruction could be suggested: the Vedic ritual shows us the theory of the application of areas in its beginnings, *as well as its original justification*. In this simple form, with areas measured by rational numbers, the application of areas would have been known by the Pythagorean sect, and used for similar ritual reasons. But while in India the theory never exceeded this embryonic stage, it was developed in Greece for its own sake, regardless of any technical or ritual use, as other embryos of mathematics were developed. There is no need to have recourse to real or supposed contacts to explain developments that could very easily have been parallel at their beginnings. In fact, a Greek legend refers to the ritual origin of a mathematical problem, the duplication of the cube: how to build a royal cubic tomb twice as big in volume than a given romb[45]. Later on, the same problem reappears in Delos, where they have to double a cubic altar; the text[46] brings all that back to the general problem of " amplify while keeping the similarity ", and then suggests technical and military applications. " Amplify while keeping the similarity ", is actually the problem of the extension of the Vedic altar[47], and also the problem solved for the rectilinear plane figures in Euclid's VI-25. The certainly ritual origin of the problem in India shows that it isn't absurd to grant an element of truth to the Greek legend about the ritual origin of an amplification keeping the similarity.

If we note strong analogies between the Vedic geometry and some fundamental problems of the *Elements*, we must note also that, at first sight, both of them equally disappoint the historian: the Vedic geometry, because many derivations, such as the theorem of the diagonal, are unknown, and the *Elements*, because they remain silent on the motives[48] that led to this grandiose edifice. All the development of Vedic geometry are motived by ritual requirements; Euclid

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44 Caveing 1997, p. 330.

45 That is a famous problem. It has exercised many Greek mathematicians, who have given solutions unacceptable with the limits of an Euclidian *doxa*: today we know that the construction isn't possible "with the ruler and the compass". We know the Greek attempts thanks to Archimedes' commentaries written up by Eutocius (Archimède 1970, p. 65)..

46 Text ascribed to Eratosthenes by Eutocius in his Archimedes' commentaries.

47 Problem similar, but much more difficult, since it is in dimension three: application of volumes and no more of areas

48 Different hypothesis on the motives of the Book II, for example, are outlined in B. Vitrac's *Euclide*, 1990, pp. 366-376.

produces the proof of all his results. Shortening to excess, we could say that the first is motivated, but without proofs, while the second proves everything, but without motives. The suggested reconstruction is maybe the begining of a solution to the problem.

Conclusion: an embryonic corpus of geometry

The most striking thing in the *ṣulbasūtras* is the harmony of the *whole*: the myth is necessary and sufficient to understand the motives of the mathematics at stake, and it is the myth that commands everything, in the slightest detail. Although the myth is teeming and obscure, its geometrical-ritual expression is transparent and well-ordered. In the geometrical action, as well as in the creation myth it actualises, everything starts from the sacrificer-demiurge: from him the units are built, and every time, he must recreate all the figures. The area and its extension provide the " model " of the fundamental energy that must spread in order to create, this energy being associated with the word that " weaves " reality.

Vedic mathematics express and strictly convey all that, as we have seen; but above all, they seem to express *only* that. I mean that, to give an account of it, there is no need to go and get explanations *outside* the Vedic mythology, except maybe for some isolated details: for example, Mānava[49] gives the formula of the volume of the parallelepiped which is used nowhere and of the circumference of the circle however useless it appears[50]. But nothing, in the texts at our disposal, allows us to think that the geometry of the *śulbasūtras*, in the specific problems they pose and solve, is a transposition of a geometry derived from surveying or building practices. There is no need either to go and get Greek, Babylonian or Egyptian influences.

The harmony of the whole, as we defined it at the beginning of this section, concerns at first sight myth and geometry, ad not geometry alone; the *śulbasūtras* appear actually as a corpus, but they don't form one in the strict sense of the term. If the applications of the theorem of the diagonal to the combining of squares follow on clearly and almost demonstratively, the theorem itself is not proved, and many results, true or false, are given without proof. There are simple visual demonstrations of the theorem of the diagonal, but nothing of it appears in the *śulbasūtras*; the Vedic

49 *M. Śulb.* 10-9.

50 *M. Śulb.*, 11-13: the circumference equals three times the diameter plus one fifth of the diameter.

practitioners could very well have known one of them and have considered it useless to put it in writing, since the object of the treatise is the sequence of building, and not the sequence of abstract properties.

But sequence there is, and, by the way of consequence, a *trend* towards a real treatise of mathematics, by force of circumstance: building is a sequence of consecutive gestures that must be described in a given order so that the practitioner can follow, and must of course produce a given result. The extension of the bird-shaped altar is produced by a sequence of actions based mainly on the theorem of the diagonal. This practical need for a sequence, as in a manual, is the origin of the remarkable structure of the *śulbasūtras*, in any case of those of Baudhāyana and Āpastamba: first, the fundamental theorem, the theorem of the diagonal, is presented, then its theoretical applications, essentially the methods for adding and subtracting squares and for transforming rectangles into squares, before turning to the practice of the building of particular altars. Thus we have a well and truly demonstrative structure that appears sometimes as such in the author's mind. In fact, after having presented two methods for constructing the square, and the theorem of the diagonal, Āpastamba proceeds: " By the understanding of these, the construction of the figures as stated[51] "; thus, he announces the future applications. Later on, in giving the method for adding two squares, he says: " A part is cut-off from the larger with the side of the smaller; the diagonal of the cut-off part combines both the squares; *this has been stated*[52] ". Later again, the transformation of a rectangle into a square, as a last operation, calls for the subtraction of two squares, and Āpastamba says again: " the removal *has been stated*[53] ". A sequence of logical operations is here definitely at work, and consciously.

These texts couldn't be reduced to simple procedures, that is to a "knack" that could be considered sufficient because it worked well. Several methods are given indeed to solve the same problem, and consequently, the Vedic geometers knew perfectly well the difference between the goal and the various ways leading towards it. The square, for example is the subject of a special "veneration", if we are to believe the numerous expounded methods of construction, as if all its mysteries were tirelessly scrutinised, rich of various aspects, and therefore *not reducible to only one of them*. Now precisely, if the mathematics were reduced here to simple procedures, the

51 *Ā. Śulb.*, 1-4.

52 *Ā. Śulb.*, 2-4.

53 .*Ā* Śulb., 2-7.

problems would be identified with a method of resolution. Similarly, Katyāyana offers three different methods for extending the bird-shaped altar.

If we believe Renou, the *sūtra* form has characteristically a phraseological condensation, the apparition of titles and conclusive formulas, cross-references to previous parts, and what he calls "rules of interpretation", " axioms which must be present in the mind of the user, so that their content could stand in at the precise suitable place[54] " The *sūtra* form is therefore a perfect framework for a corpus of geometry. Conversely, a geometer, such as Baudhāyana, could have contributed to this form; Renou thinks he was the instigator of these "rules of interpretation" and he mentions even Katyāyana, (the best mathematician of all the Vedic authors, in my opinion) as the most rigorous in the application of the *sūtra* form.

The exceptional interest of Vedic mathematics lies therefore in showing us a mathematical corpus *in gestation*, of which the motivations are clear. We have shown how they help us to look at Greek mathematics in a new light, and to wonder, on the basis of the similarity between the *śulbasūtras* and some Euclidian theorems, about a possible remote mythical-ritual origin to Greek mathematics. But there is still another reason, more profound, to do so. Mathematics, being determined quantities, are akin to ritual. As concrete ritual, they strongly oppose the poetic quality of the universal myth where beings transform themselves the one into the others: they are dispassionate, determined, absurdly meticulous. But, as abstract form, as number and expanse, as numerical combination and plane extension, they reclaim the spontaneous dialectic movement of the primitive thought, giving to the countless analogies a specific external form, for want of giving to them a real content. In the sacrifice of the goat, for example, its carcass will only be a carcass: its assimilation to the creation is a voluntary act of the mind which must forget that the goat is only a goat. On the other hand, the seven square *puruṣas* are *immediately, because they are seven*, the connection and the analogy with the seven founding figures; the area in extension is also, immediately, the abstract multiform energy. It preserves itself, changing only its form, or it increases, keeping the same form, and the endless constructions of equivalent figures are the ideal picture of the real concrete work of the multiform diffusion of the creative energy.

In such a way, mathematics, in the end, unify the myth and the rite, they give to the former the determination of the latter and to the latter the movement of the former. This fact, understood by the new mode of philosophical thinking launched for the

54 L. Renou 1963 p.178.

first time in Greece, could have been a source of the Platonic theory: mathematical ideas are "intermediate" between the world of pure thinking, metamorphosis of the myth, and the world of the practice, metamorphosis of the rite. Platonic thinking acquires in this way, if the derivation is accepted, a unexpected historical depth. It appears firmly rooted in the old primitive world, while working at the same time to cut away these roots, in order to build a *pure* thinking, but without ever renouncing this strange *intermediate* status that belongs to mathematics.

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